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RESEARCHES INTO THE WORLD OF $\kappa \to (\kappa)^\kappa$*

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In 1975 E.M. Kleinberg proved the following: Given $\kappa > \omega$ satisfying $\kappa \to (\kappa)^\kappa$, there is a sequence of cardinals $\kappa < \kappa_2 < \kappa_3 < \cdots < \kappa_\omega$ such that:

(1) κ_2 is measurable and satisfies $\kappa_2 \to (\kappa_2)^\alpha$ for all $\alpha < \kappa^+$,

(2) κ_n is a singular Jonsson cardinal of cofinality κ_2, for all $n > 2$.

This paper is a sequel to these results.

In Section 1 we construct various κ_2-additive measures μ_n on each κ_n which are Jonsson filters.

In Section 2 we prove that all ultrapowers $\kappa_n^{\kappa_m}/\mu_m$ are cofinal with κ_2 and collateral results.

And in Section 3 we define a measure ν on $[\kappa]^\kappa$ and prove that the ultrapower $\kappa^{([\kappa]^\kappa)}/\nu$ has order-type greater than κ_ω.

Modern Set Theory is the study of all possible worlds. These worlds, models of various axiom systems, are chosen for study for very many reasons: their generality, their applicability, their profundity, their plausibility, and sometimes even their whimsicality. They range in interest and usefulness from the universal to the particular, from the probable to the improbable and from the mundane to the fantastic. On the long end of each of these scales is the world of the Axiom of Determinateness.

Of all the axioms considered by set theorists in recent years, there is none to match AD's combination of profound applications, bizarre results, and disarming naturality. Its consequences in analysis alone are intriguing, as are those in descriptive set theory, but the greatest interest is provided by its effects on the structure of cardinals. These effects are both wide and deep, in that not only do they reach very high into the ordinal hierarehy, but they also provide very intimate pictures of $\aleph_1, \aleph_2, \aleph_3, \ldots$ and $\aleph_\omega$. Specifically, AD has been shown to imply the existence of a huge number of measurable cardinals and cardinals satisfying infinite-exponent partition relations [7]. $\aleph_1$ itself, under AD, is supercompact and satisfies $\aleph_1 \to (\aleph_1)^{\aleph_1}$, one of the most powerful partition properties known to man. Indeed, there is no large cardinal property of $\aleph_1$ which is known not to be implied by AD.

Interest in these results was rekindled recently when Kleinberg discovered that large parts of the theory were derivable solely from the relation $\aleph_1 \to (\aleph_1)^{\aleph_1}$, while at the same time he significantly expanded the theory with new consequences of

* Sections 1 and 3 first appeared in the author's doctoral thesis [1].

this relation. The obvious question is: How much of AD is due to $\aleph_1 \to (\aleph_1)^{\aleph_1}$ alone, and what more can be learned of AD from this property? This paper forms only a partial answer. As will presently be described, we take Kleinberg's methods a step further in several directions and develop a number of techniques which appear quite promising, but many very important questions remain unsolved. Given $\aleph_1 \to (\aleph_1)^{\aleph_1}$, there are now known to exist cardinals $\aleph_1 < \kappa_2 < \kappa_3 < \cdots < \kappa_\omega$ with the properties: κ_2 measurable, $\kappa_2 \to (\kappa_2)^\alpha$ for all $\alpha < \kappa^+$, κ_n Jonsson with cofinality κ_2 for all $n \geq 2$, κ_ω Rowbottom.

The structure of these cardinals follows from the partition relation and the fact that $\kappa_{n+1} = \kappa_n^{\aleph_1}/\mu_\omega$. In the case of AD, each κ_n is $\aleph_n$ and κ_ω is $\aleph_\omega$, but it is not likely that this can be proved from $\aleph_1 \to (\aleph_1)^{\aleph_1}$ alone. The most intriguing problem is that of finding more measurable cardinals. The Axiom of Determinateness implies, among others, that $\aleph_{\omega+1}$ is measurable, but at present, the theory of the partition relation has been unable to reach above $\aleph_\omega$ in any meaningful way.

Our starting point will be the relation $\kappa \to (\kappa)^\kappa$ and our principal tool will be the trick of substituting for p, a given member of $[\kappa]^\kappa$ ${}_\lambda p$, also in $[\kappa]^\kappa$.

0. Notation and Definitions

Our notation is reasonably standard. For any ordered set A and ordinal α, $[A]^\alpha$ will represent the set of all increasing sequences from A of length α. We will sometimes refer to a member $p \in [A]^\alpha$ as a subset of A, and sometimes as a function from α to A. At all times, however, the meaning should be clear. $[A]^{<\omega}$ will represent the set of finite subsets of A.

Definition. An ordinal γ satisfies the relation $\gamma \to (\gamma)^\alpha_B$, where α is an ordinal and B is any set, iff for all partitions $F:[\gamma]^\alpha \to B$, there is a set $p \in [\gamma]^\gamma$, such that $\overline{\overline{F''[p]^\alpha}} = 1$. The set p is said to be *homogeneous* for F. If γ does not satisfy the relation, we say $\gamma \not\to (\gamma)^\alpha_B$, and if F is a partition which has no homogeneous set, we say it is a *bad partition*. When the set B is 2, the subscript is omitted.

Definition. An ordinal γ satisfies the relation $\gamma \to [\gamma]^{<\omega}_\gamma$ iff for all partitions $F:[\gamma]^{<\omega} \to \gamma$ there is a set $p \in [\gamma]^\gamma$ such that $F''[p]^{<\omega} \neq \gamma$ γ satisfies the relation $\gamma \to [\gamma]^{<\omega}_{\alpha,\delta}$ ($\gamma \to [\gamma]^{<\omega}_{\alpha,<\delta}$ iff for all partitions $F:[\gamma]^{<\omega} \to \alpha$, there is a set $p \in [\gamma]^\gamma$ such that $\overline{\overline{F''[p]^{<\omega}}} \leq \delta$ ($\overline{\overline{F''[p]^{<\omega}}} < \delta$). As with other partition relations, p is said to be homogeneous for F. Note that the second relation defined here implies the first.

Definition. A *Jonsson cardinal* is a cardinal $\gamma > \omega$ satisfying $\gamma \to [\gamma]^{<\omega}_\gamma$.

A *Jonsson filter* U on κ is a filter such that for all partitions $F:[\gamma]^{<\omega} \to \gamma$, there is a set $B \in U$ such that B is homogeneous for F.

Definition. For γ, a cardinal, $\lambda < \gamma$, $p \subseteq \gamma$, let $(p)_\lambda$ denote the collection of sups of increasing λ-sequences from p. p is said to be λ-*closed* whenever $(p)_\lambda \subseteq p$. Note that $(p)_\lambda$ is itself λ-closed. Let μ_γ denote the mapping $\mu_\lambda : 2^\gamma \to 2$ defined by $\mu_\lambda(p) = 1$ iff p contains a λ-closed subset unbounded in λ.

A well-used concept in this paper is the following:

Definition. Given any $p \in [\kappa]^\kappa$ and $\lambda < \kappa$, we define a new element ${}_\lambda p$ of $[\kappa]^\kappa$ as follows:

$${}_\lambda p(0) = \text{the sup of the first } \lambda \text{ elements of } p,$$

$${}_\lambda p(1) = \text{the sup of the next } \lambda \text{ elements of } p,$$

and so on, and in general,

$${}_\lambda p(\alpha) = \text{the sup of the first } \lambda \text{ elements of } p \text{ greater than } \bigcup_{\beta<\alpha} {}_\lambda p(\beta).$$

Throughout this paper, κ will represent an uncountable cardinal satisfying $\kappa \to (\kappa)^\kappa$. This relation is inconsistent with well-ordered choice of length κ (see [6]). As far as is known, however, it is consistent with Dependent Choice (DC) and hence also with countable choice (AC_ω), and it will be assumed here.

The weaker relation $\kappa \to (\kappa)^{\lambda+\lambda}$, $\lambda < \kappa$ implies that μ_λ is a normal measure on κ. For any normal μ, we adopt Kleinberg's definitions and notations from [4] and [5] for the following:

(1) relations $\sim_n$ and $<_n$ on $[[\kappa]^\kappa]^{\kappa^{n-2}}$, $2 \leq n < \omega$,

(2) the functions $bk_\alpha : [\kappa]^\kappa \to [[\kappa]^\kappa]^\alpha$ and

$$sh_\alpha : [[\kappa]^\kappa]^\alpha \to {}''[\kappa]^\kappa \quad \text{for } \alpha < \kappa^+,$$

(3) the sets

$$S_n = bk_{\kappa^{n-2}}{}''[\kappa]^\kappa$$

and

$$S_n^p = bk_{\kappa^{n-2}}{}''[p]^\kappa \quad \text{for} \quad 2 \leq n < \omega, p \in [\kappa]^\kappa,$$

(4) for $H \in S_n$, $\bar{H}$ = the equivalence class of H mod$\sim_n$, and for $\alpha < \kappa$, $H_\alpha \in S_{n-1}$ is the αth block of length κ^{n-3} from H,

(5) the cardinals κ_n = the order-type of $S_n/{\sim_n} = \kappa_{n-1}^\kappa/\mu$, for $n < \omega$, and $\kappa_\omega = \bigcup_{n<\omega} \kappa_n$.

We recall that for all $p \in [\kappa]^\kappa$, $sh_\alpha(bk_\alpha(p)) = p$, and that for all $H \in [[\kappa]^\kappa]^\alpha$, and $\beta < \alpha$, $bk_\alpha(sh_\alpha(H))(\beta) \sim_2 H(\beta)$.

We note finally that for all $p \in [\kappa]^\kappa$,

$$\overline{{}_\lambda p} = \bigcup_{\delta<\lambda} bk_\alpha(p)(\delta) \quad \text{for } \lambda < \alpha.$$

For reference, we state the principal consequences of [5] we will need. Assume $\kappa \to (\kappa)^\kappa$, $\kappa > \omega$

(1) κ and κ_2 are measurable cardinals,

(2) $\kappa_2 \to (\kappa_2)^\alpha$ for all $\alpha < \kappa_2$,

(3) κ_n is a Jonsson cardinal and $cf(\kappa_n) = \kappa_2$ for $n \geqslant 2$,

(4) κ_ω is Rowbottom.

Note finally that under AD, $\aleph_1 \to (\aleph_1)^{\aleph_1}$ and for $\kappa = \aleph_1$, $\kappa_n = \aleph_n$, so that these results can be applied directly.

1. Jonsson Filters

A fundamental result of [4] is that $\kappa_2 \to (\kappa_2)^\alpha$ for any $\alpha < \kappa^+$ and hence the λ-closed, unbounded sets generate a measure on κ_2. Is the same true for κ_n? Although the answer is yes, it is not very interesting. κ_n is cofinal with κ_2, and in fact there is a cofinal κ_2-sequence in κ_n which is closed (contains all its sups). Given any subset $A \subseteq \kappa_n$, either A or its complement will contain a λ-closed subset of this cofinal sequence. This can be used to define a measure on κ_n, but it is clearly not uniform (i.e., measure-one sets may not have full cardinality) since the cofinal sequence itself will be measure-one.

Another approach is the result of examining the mechanics of proving μ_λ is a measure on κ_2 via $\kappa_2 \to (\kappa_2)^{\lambda+\lambda}$ and $\kappa \to (\kappa)^\kappa$ and generalizing. It can be seen by these methods that for $A \subseteq \kappa_2$,

$$\mu_\lambda(A) = 1 \quad \text{iff} \quad \{\overline{{}_\lambda p} \mid q \in [p]^\kappa\} \subseteq A \quad \text{for some } p \in [\kappa]^\kappa.$$

To apply this to κ_n, we use the "break" function:

Definition. For $p \in [\kappa]^\kappa$, we will denote by $\langle p \rangle_\lambda$ the set $\{{}_\lambda q \mid q \in [p]^\kappa\}$. In the case of $\lambda = \omega$, we will write $\langle p \rangle$. Note that for any λ, $[{}_\lambda p]^\kappa \subseteq \langle p \rangle_\lambda$. In view of an earlier note concerning ${}_\lambda p$, we have that $\langle p \rangle_\lambda$ is exactly $(S_2^p)_\lambda$—all the sups of λ-sequence from S_2^p.

Definition. For $2 \leq n < \omega$, we define the function $\mu_n : 2^{\kappa_n} \to 2$ by: if $A \subseteq \kappa_n$, then $\mu_n(A) = 1$ iff $(bk_{\kappa^{n-2}}{}''\langle p \rangle)/{\sim_n} \subseteq A$ for some $p \in [\kappa]^\kappa$.

Theorem 1.1. *For each n, $2 \leqslant n < \omega$, μ_n is an $\aleph_1$-additive, uniform measure on κ_n.*

Proof. First, for all $A \subseteq \kappa_n$, either $\mu_n(A) = 1$, or $\mu_n(A^c) = 1$, for we can define a partition $F : [\kappa]^\kappa \to 2$ by:

$$F(p) = 0 \quad \text{iff} \quad \overline{bk_{\kappa^{n-2}}({}_\omega p)} \in A \quad \text{for all } p \in [\kappa]^\kappa.$$

It follows that if $p \in [\kappa]^\kappa$ is any homogeneous set, then $(bk_{\kappa^{n-2}}\langle p \rangle)/{\sim_n}$ is either contained in A, or in A^c.

Next, μ_n is $\aleph_1$-additive: suppose for all k, $A_k\subseteq\kappa_n$ and $\mu_n(A_k)=1$. For each k, let p_k be such that $(bk_{\kappa^{n-2}}''\langle p_\kappa\rangle)/\sim_n\subseteq A_k$. Choose, for each k, $q_k\in[p_k]^\kappa$ so that for all α,

$$q_0(\alpha)<q_1(\alpha)<q_2(\alpha)<\cdots<q_0(\alpha+1).$$

Under these conditions, we will have

$${}_\omega q_0={}_\omega q_1={}_\omega q_2=\cdots.$$

Furthermore, for any i and j, $r\in[q_i]^\kappa$, there is a corresponding $r'\in[q_j]^\kappa$ such that ${}_\omega r={}_\omega r'$. Thus,

$$\langle q_0\rangle=\langle q_1\rangle=\langle q_2\rangle=\cdots.$$

Finally, since $\langle q_k\rangle\subseteq\langle p_k\rangle$, we then have:

$$(bk_{\kappa^{n-2}}''\langle q_0\rangle)/\sim_n\subseteq A_k\quad\text{for each } k,$$

and so

$$\mu_n\Big(\bigcup_{k<\omega}A_k\Big)=1.$$

To complete the proof, we note that μ_n is uniform, since any measure one set contains a set of the form: $(bk_{\kappa^{n-2}}''\langle p\rangle)/\sim_n$ which in turn contains the set:

$$S_n^{\omega p}/\sim_n=(bk_{\kappa^{n-2}}''[{}_\omega p]^\kappa)/\sim_n$$

since $[{}_\omega p]^\kappa\subseteq\langle p\rangle$. As we noted earlier,

$$S_n^{\omega p}/\sim_n\quad\text{is a } \kappa_n\text{-sized subset of } \kappa_n.$$

The measures μ_n can be defined using $\langle p\rangle_\lambda$ instead of $\langle p\rangle$, for any $\lambda<\kappa$, a regular cardinal. In this case, we will get distinctly different measures, but the proof of Theorem 1.1 is still valid. The important point is that if we have sequences $q_n\in[\kappa]^\kappa$, $n<\omega$, and if for all α,

$$q_0(\alpha)<q_1(\alpha)<q_2(\alpha)<\cdots<q_0(\alpha+1),$$

then we will have:

$$\langle q_0\rangle_\lambda=\langle q_1\rangle_\lambda=\langle q_2\rangle_\lambda=\cdots.$$

In the case of κ_2, the measure μ_2 is none other than μ_ω—the measure concentrating on ω-closed, unbounded sets. If we use $\langle P\rangle_\lambda$ we will, of course, obtain μ_λ on κ_2. Is it possible to define μ_κ on κ_2 in a similar manner? Since $\kappa_2\to(\kappa_2)^{\kappa+\kappa}$, we know that μ_κ *is* a measure on κ_2, and indeed it can be obtained similarly. Instead of using ${}_\lambda p$, representing the sup in κ_2 of the sequences $\{bk_\kappa(p)(\delta)\}_{\delta<\lambda}$, we use ${}_\kappa p$ to represent the sup in κ_2 of the sequences $\{bk_\kappa(p)(\delta)\}_{\delta<\kappa}$.

Definition. For any $p \in [\kappa]^\kappa$, we define ${}_\kappa p \in [\kappa]^\kappa$ by

$$ {}_\kappa p(\alpha) = \bigcup_{\delta<\alpha} bk_\kappa(p)(\delta)(\alpha) \quad \text{for each } \alpha < \kappa. $$

Note that

$$ \overline{{}_\kappa p} = \bigcup_{\delta<\kappa} \overline{bk_\kappa(p)(\delta)} $$

in κ_2, since for any $\delta < \kappa$,

$$ \{\alpha \mid {}_\kappa p(\alpha) > bk_\kappa(p)(\delta)(\alpha)\} $$

is $\kappa - \delta$ and clearly a measure-one set. Furthermore, if $q \in [\kappa]^\kappa$ is greater than $bk_\kappa(p)(\delta)$ for all $\delta < \kappa$, the normality of μ guarantees that ${}_\kappa p \leqslant q$ as follows: for each δ, let $A_\delta = \{\alpha \mid bk_\kappa(p)(\delta)(\alpha) < q(\alpha)\}$ and let

$$ A = \mathop{\Delta}_{\delta<\kappa} A_\delta = \{\alpha \mid \alpha \in A_\delta \quad \text{for all } \delta < \alpha\}. $$

Since μ is normal, $\mu(A) = 1$, and $\alpha \in A$ implies ${}_\kappa p(\alpha) \leqslant q(\alpha)$.

$\langle p \rangle_\kappa$ is defined in the same manner as before, and it can then be proved that:

Lemma 1.2. *For any set $A \subseteq \kappa_2$, $\mu_\kappa(A) = 1$ iff $\langle p \rangle_\kappa / \sim_2 \subseteq A$ for some $p \in [\kappa]^\kappa$.*

Proof. It is easy to see, as in the proof of Theorem 1.1, that given A, we can find a p such that either $\langle p \rangle_\kappa / \sim_2 \subseteq A$ or $\langle p \rangle_\kappa / \sim_2 \subseteq A^c$. Thus, it only remains to be shown that if $\mu_\kappa(A) = 1$, there cannot be such a $p \in [\kappa]^\kappa$ with $\langle p \rangle_\kappa / \sim_2 \subseteq A^c$. Suppose, however, that $\mu_\kappa(A) = 1$, and $\langle p \rangle_\kappa / \sim_2 \subseteq A^c$. Let $B \subseteq A$ be a κ-closed, unbounded subset of κ_2.

Definition. For any $q \in [\kappa]^\kappa$, let q^0 and q^1 be respectively the range of q on the even and the odd ordinals, in our notation,

$$ q^0 = bk_2(q)(0) \quad \text{and} \quad q^1 = bk_2(q)(1). $$

Now, let $F : [\kappa]^\kappa \to 2$ be the following partition:

$$ F(q) = 0 \quad \text{iff there is an element of B between } \overline{q^0} \text{ and } \overline{q^1}. $$

Let $q \in [\kappa]^\kappa$ be homogeneous for F. $F``[q]^\kappa$ must equal 0, for we can easily find $r, s \in S_2^q$ such that there is an element of B between them, since $S_2^q / \sim_2$ is a set of size κ_2. These sequences can then be shuffled together: $t = sh_2(r, s) \in [q]^\kappa$, and then since $t^0 = r$ and $t^1 = s$, we have $F(t) = 0$.

Finally, choose $q' \in [q]^\kappa$ and $p' \in [p]^\kappa$ so that for all $\alpha < \kappa$

$$ q'(\alpha) < p'(\alpha) < q'(\alpha + 1), $$

and consider the sequences ${}_\kappa q'$ and ${}_\kappa p'$. They are not equal at all ordinals α, but they clearly are for all limit ordinals α, by the definition of ${}_\kappa q'$ and ${}_\kappa p'$, and since $\mu(\{\text{all limit ordinals}\})=1$, ${}_\kappa q' \sim_{2\kappa} p' \cdot {}_\kappa q'$, however, is the sup of the $\{bk_\kappa(q')(\delta)\}_{\delta<\kappa}$, and between any two of these κ sequences there is an element of B (any two of them can be shuffled together as r and s were above, and since the range of F on $[q]^\kappa$ is 0, there is an ordinal from B between them). Thus ${}_\kappa q'$ is also the limit of a κ-sequence from B and so is in B. This puts ${}_\kappa p'$ in B a contradiction, since we assumed at the start that $\langle p\rangle_\kappa/\sim_2 \subseteq A^c$.

Measures similar to μ_κ, based on $\langle p\rangle_\kappa$ can also be defined on κ_n.

Notice that in proving Theorem 2.1 we have not proved that μ_n is κ_n-additive. It can't be, of course, since κ_n is singular. It can, however, be κ_2-additive, and we will prove that it is as a corollary of the following theorem:

Theorem 1.3. *For each n, $2<n<\omega$, the subsets of κ_n of μ_n-measure one form a Jonsson filter. Further, they form a filter for the relation:*

$$\kappa_n \to [\kappa_n]^{<\omega}_{\gamma,\omega} \quad \text{for all } \gamma<\kappa_2.$$

Proof. The proof of this theorem consists of examining the proof of Lemma G.2 in [5] and in place of the functions bk_α, using bk^*_α defined by $bk^*_\alpha(p)=bk_\alpha({}_\omega p)$. The result follows from Kleinberg's methods and the following lemma:

Lemma 1.4. *Given $\gamma<\kappa_2$, $2<n<\omega$, $k<\omega$, and $F:[\kappa_n]^k\to\gamma$, there is a partition $G:[\kappa]^\kappa\to 2$ such that if p is homogeneous for G, then*

$$Y=(bk^*_{\kappa_{n-2}}{}''[p]^\kappa)/\sim_n$$

is homogeneous for F, in that $F''[Y]^k$ is finite.

Theorem 1.5. *For all n, $2<n<\omega$, μ_n is κ_2-additive.*

Proof. Suppose that for some n, $\gamma<\kappa_2$, $\{A_\alpha\}_{\alpha<\gamma}$ is a disjoint collection of subsets of κ_2 with $\kappa_n=\bigcup_{\alpha<\gamma}A$. Let $F:[\kappa_n]\to\gamma$ be defined as:

$$F(\alpha)=\beta \quad \text{iff} \quad \alpha\in A_\beta.$$

Since μ_n defines a filter for the partition $\kappa_n\to[\kappa_n]^{<\omega}_{\gamma,\omega}$, there is a set $X\in[\kappa_n]^{\kappa_n}$, such that $\mu_n(X)=1$ and $F''[X]^1\leq\omega$. Thus X is contained in a countable subset of the A_α. By the countable additivity of μ_n, one of these must have μ_n-measure-one.

Corollary 1.6. AD+DC$\vdash$*For each n, $2<n\leq\omega$, $\aleph_n$ is a singular Jonsson cardinal with an $\aleph_2$-additive Jonsson filter.*

2. Ultrapowers of the κ_n

Looking at the sequence of cardinals: κ, $\kappa^\kappa=\kappa_2$, $\kappa_2^\kappa=\kappa_3$, $\kappa_3^\kappa=\kappa_4,\ldots$ one notes that the nicest properties of these drop off after κ_2. That is, κ and κ_2 are measurable and satisfy infinite-exponent partition relations, but κ_3 is not even regular. One is tempted to suggest that the sequence was continued in the wrong manner, that we should have proceeded:

$$\kappa, \quad \kappa^\kappa = \kappa_2, \quad \kappa_2^{\kappa_2} = \kappa_3', \quad \kappa_3^{\kappa_3'} = \kappa_4.$$

In trying to examine κ_3', there is a reasonable-looking approach, namely given $p \in [\kappa]^\kappa$, $[p]^\kappa$ is a κ_2-sized subset of κ_2 and represents an element in $\kappa_3' = [\kappa_2]^{\kappa_2}/\mu_2$. Let $g : [\kappa]^\kappa \to \kappa_3'$ be this function, i.e., $g(p) =$ the equivalence class mod μ_2 of $[p]^\kappa/\sim_2$. A measure on κ_3' can then be defined in the usual way, given $A \in \kappa_3$, let $F : [\kappa]^\kappa \to 2$ be the partition: $F(p) = 0$ iff $g({}_\omega p) \in A$. If p is homogeneous for F, then either $g''\langle p\rangle$ is contained in A or A^c. If we define a set A to be measure one whenever $g''\langle p\rangle \subseteq A$ for some $p \in [\kappa]^\kappa$, it can easily be shown to be at least $\aleph_1$-additive. One thing is wrong, however. It can be shown in certain cases (depending on the measure μ on κ) that for any $p, q \in [\kappa]^\kappa$, $p \sim_2 q$ implies $g(p) = g(q)$. Thus $g''[\kappa]^\kappa$ has cardinality at most κ_2. The measure we are defining is concentrating on a set of size κ_2. If we still expect that κ_3 will prove to be measurable, we must hope that $g''[\kappa]^\kappa$ is not cofinal in κ_3. Unfortunately, it is.

We are guided in these matters by two results of Kunen. Under AD, Kunen has shown that:

(1) $\aleph_2^{\aleph_2}/\mu$ is $\aleph_3$ and

(2) for all measures μ on $\aleph_1$,

$\aleph_1^{\aleph_1}/\mu$ is $\aleph_n$ for some n.

We cannot as yet match these results. We can show, though, that if the measure μ on κ is μ_λ, for some cardinal $\omega \leq \lambda < \kappa$, then for all m and n, the ultrapower: $\kappa_n^{\kappa_m}/\mu_m$ has cofinality κ_2.

Due to a number of structure lemmas, this section is rather long. In outline, our first target is to show cf $(\kappa_2^{\kappa_2}/\mu_\lambda) = \kappa_2$ for all infinite cardinals $\lambda \leq \kappa$. Next, we reduce the problem of $\kappa_n^{\kappa_m}/\mu_m$ by showing that $\mathrm{cf}(\kappa_n^{\kappa_m}/\mu_m) = \mathrm{cf}(\kappa_2^{\kappa_m}/\mu_m)$. To attack this last problem, we introduce Kleinberg's notion of "i-interlaced pairs" from κ_m, establishing finally that:

$$\kappa_2^{\kappa_m}/\mu_m = \kappa_2^{\kappa_2}/\mu_\kappa.$$

We will conclude by constructing a κ_3 cofinal sequence in $\kappa_2^{\kappa_2}/\mu_2$, one which some day may prove to be all of $\kappa_2^{\kappa_2}/\mu_2$.

We begin with a lemma:

Lemma 2.1. *For any $r, p \in [\kappa]^\kappa$, the* rth *element of* $[p]^\kappa/\sim_2$ *is* $\overline{p \circ r}$.

Proof. Consider the map $H : [\kappa]^\kappa \to [p]^\kappa$ defined by:

$$h(\bar{r}) = \overline{p \circ r}.$$

This is easily seen to be both well-defined and order-preserving. It is also onto, for if $q \in [p]^\kappa$, then clearly $q = p \circ r$ for some $r \in [\kappa]^\kappa$.

Theorem 2.2. *If the measure μ on κ is μ_λ for some $\omega \leq \lambda < \kappa$, and $\langle\ \rangle_\lambda$ is used to define μ_2, then the cofinality of $\kappa_2^{\kappa_2}/\mu_2$ is κ_2.*

Proof. Let $g:[\kappa]^\kappa \to [\kappa_2]^{\kappa_2}/\mu_2$ be the function discussed earlier, and suppose that $p, q \in [\kappa]^\kappa$ are such that $p \sim_2 q$. Since $\mu_\lambda(\{\alpha \mid p(\alpha) = q(\alpha)\}) = 1$, let $s \subseteq \{\alpha \mid p(\alpha) = q(\alpha)\}$ be a λ-closed, unbounded subset, and consider

$$B = \langle s \rangle_\lambda / \sim_2.$$

We have $\mu_2(B) = 1$, by the definition of μ_2. We also have that for any $\alpha = \overline{_\lambda r} \in B$, $r \in [s]^\kappa$, the αth elements of $[p]^\kappa$ and $[q]^\kappa$ are equal as follows: by Lemma 2.1, the αth elements of $[p]^\kappa$ and $[q]^\kappa$ are respectively $p \circ {_\lambda r}$ and $q \circ {_\lambda r}$, but p and q are equal on the range of $_\lambda r \subseteq s$. Thus, $p \sim_2 q$ implies $g(p) = g(q)$, since $[p]^\kappa$ and $[q]^\kappa$ are equal (as elements of $[\kappa_2]^{\kappa_2}$) on a set of measure one. Similarly, $p <_2 q$ implies $g(p) < g(q)$, so that g maps κ_2 order-preservingly into $\kappa_2^{\kappa_2}/\mu_2$.

Claim. *g is unbounded.*

Proof of claim. Suppose H is any element of $[\kappa_2]^{\kappa_2}$. Let $F:[\kappa]^\kappa \to 2$ be the following partition: $F(p) = 0$ iff $\overline{p^1} > H(\overline{p^0})$ (this notation was introduced in the proof of Lemma 1.2). Let $p \in [\kappa]^\kappa$ be homogeneous for F. Since $[p]^\kappa$ is unbounded in κ_2, let $q \in [p]^\kappa$ be such that $\bar{q} > H(\bar{p})$, and let $r = sh_2(p, q)$. $r \in [p]^\kappa$, and $r^0 \sim_2 p$ and $r^1 \sim_2 q$, so $F(r) = 0$, hence $F''[p]^\kappa = \{0\}$. This is enough to guarantee that $g(p^1)$ is greater than H in the ultraproduct $\kappa_2^{\kappa_2}/\mu_2$, for if $\alpha < \kappa_2$, then $\alpha = \bar{r}$ for some $r \in [\kappa]^\kappa$ and so the αth elements of p^0 and p^1 are respectively $\overline{p^0 \circ r}$ and $\overline{p^1 \circ r}$. Let $q = sh_2(p^0 \circ r, p^1 \circ r)$. Then since $F(q) = 0$, we must have $\overline{p^1 \circ r} > H(\overline{p^0 \circ r}) \geq H(\alpha)$ and hence the αth element of p^1 is strictly greater than the αth element of H.

An important corollary of this is:

Corollary 2.3. *Under the assumptions of Theorem 2.2, $\kappa_2 \not\to (\kappa_2)^{\kappa_2}$.*

Proof. From Kleinberg's work, $\kappa_2 \to (\kappa_2)^{\kappa_2}$ would imply that $\kappa_2^{\kappa_2}/\mu_2$ is measurable.

Theorem 2.2 handles the ultrapower of κ_2 under the measure μ_2, which as we noted earlier is μ_λ when $\langle\ \rangle_\lambda$ is used in its definition, and μ_λ is used as the measure on κ. A similar result holds for the measure μ_κ:

Theorem 2.4. *Under the same hypotheses as Theorem 3.2, $cf(\kappa_2^{\kappa_2}/\mu_\kappa) = \kappa_2$.*

Proof. Once again, let g be as defined earlier except this time the range is $\kappa_2^{\kappa_2}/\mu_\kappa$, and let $p \sim_2 q$ be elements of $[\kappa]^\kappa$. Let $s \subseteq \{\alpha \mid p(\alpha) = q(\alpha)\}$ be a λ-closed, unbounded subset of κ, and let $B = \langle s \rangle_\kappa$. We have $\mu_\kappa(B) = 1$ by Lemma 1.2. We also have that for any $\alpha \in B$, the αth elements of $[p]^\kappa/{\sim_2}$ and $[q]^\kappa/{\sim_2}$ are equal as follows: since $\alpha = \overline{{}_\kappa r}$ for some $r \in [s]^\kappa$, the αth elements of $[p]^\kappa/{\sim_2}$ and $[q]^\kappa/{\sim_2}$ are respectively: $p \circ {}_\kappa r$ and $q \circ {}_\kappa r$. But for all ordinals δ of cofinality λ, ${}_\kappa r(\delta)$ is a δ-sup from s, hence in s, so

$$p \circ {}_\kappa r(\delta) = q \circ {}_\kappa r(\delta).$$

Thus $p \circ {}_\kappa r \sim_2 q \circ {}_\kappa r$, since they agree on a μ_λ-measure one set (the set of ordinals of cofinality λ), and thus $g(p) = g(q)$, since $[p]^\kappa/{\sim_2}$ and $[q]^\kappa/{\sim_2}$ agree on a set of μ_κ-measure one (i.e., B). Similarly $p <_2 q$ implies $g(p) < g(q)$ and so g generates an increasing function from κ_2 to $\kappa_2^{\kappa_2}/\mu_\kappa$. The proof that g is unbounded is identical to that in Theorem 2.2.

The remaining ultrapowers, $\kappa_n^{\kappa_m}/\mu_m$ involve considerable subtleties. One simplification is straight-forward, however:

Lemma 2.5. *For all $2 \leqslant n$, $m < \omega$, the cofinality of $\kappa_n^{\kappa_m}/\mu_m$ is the same as that of $\kappa_2^{\kappa_m}/\mu_m$.*

Proof. Let A be a cofinal subset of κ_n of order-type κ_2. Then A^{κ_m}/μ_m is a subset of $\kappa_n^{\kappa_m}/\mu_m$ of type $\kappa_2^{\kappa_m}/\mu_m$. Furthermore, it is cofinal, for if $f \in [\kappa_n]^{\kappa_m}$, we may define $g \in [A]^{\kappa_m}$ by:

$$g(\alpha) = \text{the least element of } A - f(\alpha) \quad \text{for all } \alpha < \kappa_m.$$

Then f is clearly less than or equal to g in $\kappa_n^{\kappa_m}/\mu_m$.

It only remains now for us to find the cofinality of $\kappa_2^{\kappa_n}/\mu_n$, for $n \geqslant 3$. We will do more: we will in fact show that $\kappa_2^{\kappa_n}/\mu_n = \kappa_2^{\kappa_2}/\mu_\kappa$, but this will not be easy. To proceed further we must introduce still more of Kleinberg's machinery. The difficulty we will have, and the same difficulty that prevents $\kappa_3, \kappa_4, \ldots$ from having nicer properties is that there are distinctly different kinds of pairs of ordinals less than these cardinals. View κ_3, for a moment, as $\kappa_2^{\kappa_2}/\mu$. Given two sequences $p, q \in [\kappa_2]^\kappa$, it could be that $\bigcup p = \bigcup q$, or it could be that $\bigcup p \neq \bigcup q$. These two cases are fundamentally different, as we shall see.

Definition. For $n > 2$, H, $G \in S_n$, we say H and G are 0-*interlaced* iff

$$\bigcup_{\alpha<\kappa} \overline{H_\alpha} \neq \bigcup_{\alpha<\kappa} \overline{G_\alpha}$$

(where these sups are taken in κ_{n-1}). For $n = 2$, we define all pairs of elements of S to be 0-interlaced. For $n > 2$, H, $G \in S_n$, if H and G are not 0-interlaced, we say they are $i+1$-*interlaced* whenever $\mu(\{\alpha \mid H_\alpha \text{ and } G_\alpha \text{ are } i\text{-interlaced}\}) = 1$. Note that by definition, the components of any member of S_n, $n > 2$, are 0-interlaced members of S_{n-1}.

Lemma 2.6. *The definition of i-interlacing is well-defined on equivalence classes of $\sim_n$, for all n.*

This is lemma D.1 of [5].

The relation of being i-interlaced is in general *not* an equivalence relation on S_n, although Kleinberg shows this to be the case when $i=n-2$. The following relation is an equivalence relation:

Definition. For $n>2$, $\alpha,\beta\in\kappa_n$, we define $\alpha\sim_n^*\beta$ iff α and β are not 0-interlaced.

Lemma 2.7. *$\sim_n^*$ is an equivalence relation on κ_n, and $<_n$ induces a well-ordering on the equivalence classes. Furthermore, the order-type of this ordering is κ_2.*

Proof. That $\sim_n^*$ is an equivalence relation follows directly from the definition of 0-interlacing. Let $<_n^*$ be defined on the equivalence classes by: For any two classes $A\neq B$, $\alpha\in A$, $\beta\in B$,

$$A<_n^* B \quad \text{iff } \alpha<_n\beta.$$

Then for any a, $\beta\in\kappa_n$, if $H, G\in S_n$ are such that $\bar H=\alpha$ and $\bar G=\beta$, it follows that the class of α is $<_n^*$ the class of β iff

$$\bigcup_{\delta<\kappa}\overline{H_\delta}<\bigcup_{\delta<\kappa}\overline{G_\delta}.$$

This establishes that $<_n$ well-orders the equivalence classes of $\sim_n^*$.

Lastly, we prove by induction that $S_n/\sim_n^*$ has order-type κ_2. For $n=3$, we see that the equivalence class of an element H of S_3 is characterized by the ordinal $\alpha=\bigcup_{\delta<\kappa}\overline{H_\delta}<\kappa_2$. Since there are only κ_2 such ordinals, there are at most κ_2-many classes. For similar reasons, every initial segment of $<_3^*$ has cardinality less than κ_2, and thus the type of $S_3/\sim_3^*$ is $\leqslant\kappa_2$. To complete the equality, note that if the order-type were less than κ_2, then the collection of sups $\alpha=\bigcup_{\delta<\kappa}\overline{H_\delta}$ for $H\in S_3$ would be bounded, which isn't the case (if β were a bound, let $\bar p\in[\kappa]^\kappa$ be such that $\bar p>\beta$, and let $H=bk_\kappa(p)$).

Given $n\geqslant 3$, let B_n be the set consisting of the $<_n$-least ordinal in each equivalence class, and assume that the order-type of B_{n-1} is κ_2. Now for any $H\in S_n$, the equivalence class of H is characterized by $\alpha=\bigcup_{\delta<\kappa}\overline{H_\delta}<\kappa_{n-1}$. The $\overline{H_\delta}$, however, are all 0-interlaced ordinals below κ_{n-1}, and so between any two of them lies an element of B_{n-1}. Thus α is a limit point of B_{n-1} and since there are only κ_2 such points, we have $\bar B_n\leqslant\kappa_2$. Finally, B_n is unbounded in κ_n completing the proof, since if $H\in S_n$, and $\bar p=\bigcup_{\delta<\kappa^{n-2}}\overline{H(\delta)}$, then H is 0-interlaced with $bk_{\kappa^{n-2}}(p)$ and so $\bar H$ is less than an element of B_n.

The problem of interlacing demonstrates why, for example, $\kappa_3\not\to(\kappa_3)^2$ namely the partition: $F:[\kappa_3]^2 \to 2$ defined by

$$F(\alpha, \beta) = 0 \quad \text{iff } \alpha \text{ and } \beta \text{ are 0-interlaced,}$$

can have no homogeneous set. If $X \subseteq \kappa_3$ is such that all elements of X are 0-interlaced, then the preceding lemma shows that $\bar{X} \leq \kappa_2$. On the other hand, if all the elements of X are 1-interlaced, one can see that X is bounded below κ_3. Perhaps the most astonishing property of the singular cardinals κ_n, $n > 2$, is that this is the worst partition, that is:

Lemma 2.8. (Kleinberg). *For all $n > 2$, $\lambda < \kappa_2$, and partitions $F:[\kappa_n]^2 \to \lambda$ there is a set $p \in [\kappa]^\kappa$ $X = S_n^p/{\sim_n}$, such that F is constant on all pairs from X of the same interlacing. To be precise, if α, β, λ, $\delta \in X$ and $i \leq n-2$ is such that the pair α and β and the pair λ and δ are both i-interlaced, then $F(\alpha, \beta) = F(\lambda, \delta)$.*

In the same manner as Theorem 1.3 we have:

Lemma 2.9. *For all $n < 2$, $\gamma < \kappa_2$, $F:[\kappa_n]^2 \to \gamma$, there is a subset X of κ_n of μ_n-measure one such that F is constant on all pairs from X of the same interlacing.*

We will need one more lemma for our proof:

Lemma 2.10. *For all $2 < n < \omega$, $0 < i \leq n-2$ and $p \in [\kappa]^\kappa$, there is a sequence of length κ_2 from $S_n^p/{\sim_n}$, all pairs from which are i-interlaced.*

Proof. For any n, i, $H \in S_n$, we will define a map $F_{i,n}^H:\kappa_2 \to S_n/{\sim_n}$ such that:

(1) $F_{i,n}^H$ is order-preserving;

(2) all pairs in the range of F are i-interlaced, and

(3) for any α, $F_{i,n}^H(\alpha)$ is a subsequence of H (when viewed as a κ^{n-2}-sequence from $[\kappa]^\kappa$).

Setting $H = bk_{\kappa^{n-2}}(p)$ then completes the proof.

First, we define $F_{1,n}^H(\bar{q})$ for $q \in [\kappa]^\kappa$ by:

$$F_{1,n}^H(\bar{q})_\alpha = \overline{H_{q(\alpha)}}.$$

Note first that the function is well-defined, for if for measure-one many α, $q_1(\alpha) = q_2(\alpha)$, then for the same set of ordinals,

$$F_{1,n}^H(\overline{q_1})_\alpha = F_{1,n}^H(\overline{q_2})_\alpha.$$

Similarly, $F_{1,n}^H$ is order-preserving.

If $q_1 < q_2$, $F_{1,n}^H(\overline{q_1})_\alpha$ and $F_{1,n}^H(\overline{q_2})_\alpha$ are not 0-interlaced, for

$$\bigcup_{\alpha<\kappa} F_{1,n}^H(\overline{q_1})_\alpha = \bigcup_{\alpha<\kappa} \bar{H}_\alpha = \bigcup_{\alpha<\kappa} F_{1,n}^H(\overline{q_2})_\alpha.$$

But for any α such that $q_1(\alpha) < q_2(\alpha)$, $F_{1,n}^H(\overline{q_1})_\alpha$ and $F_{1,n}^H(\overline{q_2})_\alpha$ are 0-interlaced, being distinct components of H, and so $F_{1,n}^H(\overline{q_1})_\alpha$ and $F_{1,n}^H(\overline{q_2})_\alpha$ are 1-interlaced.

Finally, it should be clear that $F^H_{1,n}(\alpha)$ is a subsequence of H.

Now for $i>1$, we define:

$$F^H_{i,n}(\bar{q})_\alpha = F^{H_\alpha}_{i-1,n-1}(\bar{q}) \quad \text{for all } q.$$

That $F^H_{i,n}$ is well-defined and order-preserving is trivial by induction. Property (3) above is likewise a consequence of the induction hypothesis, and we are left with only (2) to prove.

Suppose $\alpha_1 < \alpha_2$. Then for any $q_1, q_2 \in [\kappa]^\kappa$,

$$F^{H_{\alpha_1}}_{i-1,n-1}(\overline{q_1}) < F^{H_{\alpha_2}}_{i-1,n-1}(\overline{q_2}) \quad \text{by (3).}$$

Thus, if $q_1 \not\sim_2 q_2$,

$$\overline{\bigcup_{\alpha<\kappa} F^H_{i,n}(q_1)_\alpha} = \overline{\bigcup_{\alpha<\kappa} F^H_{i,n}(q_2)_\alpha},$$

so $F^H_{i,n}\overline{(q_1)}$ and $F^H_{i,n}\overline{(q_2)}$ are not 0-interlaced. On the other hand, for all α, $F^{H_\alpha}_{i-1,n-1}\overline{(q_1)}$ and $F^{H_\alpha}_{i-1,n-1}\overline{(q_2)}$ are i-1-interlaced, so $F^H_{i,n}\overline{(q_1)}$ and $F^H_{i,n}\overline{(q_2)}$ are i-interlaced.

Theorem 2.11. *For all $n>2$, the order-type of $\kappa_2^{\kappa_n}/\mu_n$ is equal to that of $\kappa_2^{\kappa_2}/\mu_\kappa$, assuming as before that the measure μ on κ is μ_λ for some $\omega \leq \lambda < \kappa$.*

Proof. For convenience, we will assume the measure μ on κ is μ_ω. Given $f:\kappa_n \to \kappa_2$, our plan is to show that f is equivalent, mod μ_n to a function g which is constant on equivalence classes of $\sim^*_n$. Since there are exactly κ_2 equivalence classes, by Lemma 2.7, we will then have that the order-type of $\kappa_2^{\kappa_n}/\mu_2$ must be the same as that of $\kappa_2^{\kappa_2}/\mu^*$—where μ^* is the measure induced on κ_2 as follows: for $\alpha < \kappa_n$, let $s(\alpha)=\beta$, where α is in the βth equivalence class of $\sim^*_n$. Then

$$\mu^*(A) = \mu_n(s^{-1}(A)) \quad \text{for all } A \subseteq \kappa_n$$

(i.e., $\mu^*(A) = \mu_n(\bigcup_{\alpha\in A}$ the αth equivalence class of $\sim^*_n$)). We will then show that μ^* and μ_κ are the same, and the theorem will follow from Theorem 2.4.

Now, given a function f from κ_n to κ_2, let $F:[\kappa_n]^2 \to 3$ be defined by:

$$F(\alpha,\beta) = \begin{cases} 0 & \text{if } f(\alpha)<f(\beta), \\ 1 & \text{if } f(\alpha)=f(\beta), \\ 2 & \text{if } f(\alpha)>f(\beta). \end{cases}$$

By Lemma 2.9, there is a set $X \subseteq \kappa_n$, with $\mu_n(X)=1$, such that F is constant on all pairs of the same interlacing. In view of Lemmas 2.7 and 2.10, $F(\alpha,\beta)$ can never be 2 for $\alpha, \beta \in X$, regardless of the interlacing, for this would lead to an infinite descending sequence.

Claim. *For $i>0$, the range of F on the i-interlaced pairs from X is 1 (and hence f is constant on the equivalence classes of $\sim^*_n$ on a measure-one set).*

Proof of claim. Suppose for some $i > 0$, the range of F on the i-interlaced pairs from X is 0. Since $\mu_n(X) = 1$, there is a $p \in [\kappa]^\kappa$ such that

$$bk_{\kappa^{n-2}}''\langle p\rangle/\sim_n \subseteq X$$

and more particularly, $S_n^{\omega p}/\sim_n \subseteq X$. If we let $H = bk_{\kappa^{n-2}}(_\omega p) \in X$, we have:

$$(F_{i,n}^{H}{}''\kappa_2)/\sim_2 \subseteq S_n^{\omega p}/\sim_n \subseteq X$$

by property (3) in the proof of Lemma 2.10. $(F_{i,n}^{H}{}''\kappa_2)/\sim_n$ is then a sequence of length κ_2 in X, all i-interlaced, and hence a sequence on which f is strictly increasing. This is impossible, for we can choose a $\beta \in \underset{=}{X}$, greater than the κ_2-sequence and 0-interlaced with all its members (since $\overset{=}{X} = \kappa_n$ and the κ_2-sequence falls into a single equivalence class of $\sim_n^*$ which must be bounded by Lemma 2.7). Then by homogeneity, $f(\beta)$ is greater than or equal to f applied to any element of the sequence. Since $f(\beta) < \kappa_2$, this shows that f cannot be strictly increasing on the sequence, proving the claim.

In view of the above facts, we have only to prove that μ^* and μ_κ are the same.

Suppose $A \subseteq \kappa_2$ and $\mu^*(A) = 1$, with $\mu_\kappa(A) = 0$. Let $B \subseteq A^c$ be κ-closed and unbounded. We will find an ordinal in $B \cap A$ to form a contradiction.

By the definition of μ^*, there is a $p \in [\kappa]^\kappa$ such that

$$s''(bk_{\kappa^{n-2}}\langle p\rangle)/\sim_n \subseteq A.$$

We define a partition $G : [p]^\kappa \to 2$ as follows: for any $q \in [p]^\kappa$, break $_\omega q$ into two consecutive κ^{n-2}-sequences forming two elements of S_n, and let $G(q) = 0$ if and only if there is an element of B between them. Let $r \in [p]^\kappa$ be homogeneous for G. The range of G on $[r]^\kappa$ must be 0, for we can find two elements $H_1 < H_2$ of $S_n^{\omega r}$, 0-interlaced such that there is an element of B between $s(H_1)$ and $s(H_2)$, since $s''(S_n^{\omega r})$ and B are both unbounded. H_1 and H_2 can then be shuffled together to yield $t \in [_\omega r]^\kappa$, $t = {_\omega q}$, $q \in [r]^\kappa$ such that breaking $_\omega q$ into the two consecutive κ^{n-2} sequences gives us H_1 and H_2 and so $G(q) = 0$.

Now, let us define $H \in S_n$ by:

$$H_\alpha = [bk_{\kappa^{n-1}}(_\omega r)_\alpha]_\alpha \quad \text{for all } \alpha < \kappa.$$

It is easily verified that H is a member of $S_n^{\omega r}$, that it is an increasing sequence from $[_\omega r]^\kappa$ of length κ^{n-2}, and so

$$s(\bar{H}) \in s''(bk_{\kappa^{n-2}}\langle p\rangle)/\sim_n \subseteq A.$$

However, $s(\bar{H}) \in B$ as well as follows: $\bar{H}$ is the sup (in κ_n) of the κ-sequence:

$$\{bk_{\kappa^{n-1}}(_\omega r)_\alpha\}_{\alpha<\kappa}$$

For each α, let $H^\alpha = bk_{\kappa^{n-1}}(_\omega r)_\alpha$. Since the $\{H^\alpha\}$ are 0-interlaced, the collection: $\{s(\overline{H^\alpha})\}_{\alpha<\kappa}$ is also a κ-sequence, and it is easy to see that $s(\bar{H})$ is the sup.

Finally, between any two elements of the sequence, there is an element of B, by the homogeneity of r (for any $\alpha_1 < \alpha_2$, $bk_{\kappa^{n-1}}(_\omega r)_{\alpha_1}$ and $bk_{\kappa^{n-1}}(_\omega r)_{\alpha_2}$ can be shuffled together, and then G can be applied, etc.) and so $s(\bar{H})$ is the limit of a κ-sequence from B, and is therefore in B. This completes the contradiction and the proof of the theorem. Note that while we have used ${}_\omega p$ and $\langle\,\rangle$ here, ${}_\lambda p$ and $\langle\,\rangle_\lambda$ can be substituted with no difficulty.

To sum up:

Theorem 2.12. *If for some $\omega \leq \lambda < \kappa$, the measure μ on κ is μ_λ and $\langle\,\rangle_\lambda$ is used to define μ_2, then for all n, m, $2 \leq n, m < \omega$, $cf(\kappa_n^{\kappa_m}/\mu_m) = \kappa_2$.*

It is tempting to try to remove the restriction on μ, but it may be difficult. It is used in the proofs of Theorems 2.2 and 2.4 to get a subset of a measure-one set with a certain closure property, and it is not immediately clear how this is to be done with an arbitrary normal measure.

To close this section, we prove a theorem which provides a slightly more detailed picture of $\kappa_2^{\kappa_2}/\mu_2$.

Theorem 2.13. *If for some $\omega \leq \lambda < \kappa$ the measure μ on κ is μ_λ and $\langle\,\rangle_\lambda$ is used to define μ_2, then $\kappa_3 \leq \kappa_2^{\kappa_2}/\mu_2$.*

Proof. We will define the sequence $\{\lambda_\delta\}_{\delta<\kappa_3}$ as follows: for $\delta = \bar{H}$, $H \in S_3$, λ_δ will be the equivalence class mod μ_2 of the element G of $[\kappa_2]^{\kappa_2}$ defined by:

$$G(\alpha) = \bar{q} \quad \text{for } \alpha = \bar{p},\ p \in [\kappa]^\kappa,$$

where

$$q(\beta) = H_\beta(p(\beta)) \quad \text{for all } \beta < \kappa.$$

Claim. *γ_δ is well-defined.*

We must show that γ_δ is independent of the choices: p for $\bar{p} = \alpha$ and H for $\bar{H} = \delta$. First note that if $p \sim_2 q$, then $\mu(\{\beta \mid p(\beta) = q(\beta)\}) = 1$ and so $\mu(\{\beta \mid H_\beta(p(\beta)) = H_\beta(q(\beta))\}) = 1$. Thus the definition of $G(\alpha)$ is independent of the choice $\bar{p} = \alpha$.

Next, suppose $H_1 \sim_3 H_2$, and let $G_1, G_2 \in [\kappa_2]^{\kappa_2}$ be defined respectively from H_1 and H_2 as above. Let $A = \{\beta \mid H_{1_\beta} \sim_2 H_{2_\beta}\}$, and for each $\beta \in A$, let

$$B_\beta = \{\eta \mid H_{1_\beta}(\eta) = H_{2_\beta}(\eta)\}.$$

Let $B = \Delta B_\beta = \{\eta \mid \eta \in B_\beta \text{ for } \beta < \eta\}$ and let $v \subseteq B$ be a λ-closed unbounded subset (since $\mu_\lambda(B) = 1$). Finally, let $D = \overline{\langle v \rangle_\lambda}/\sim_2$, $\mu_2(D) = 1$. We will show that for all $\alpha \in D$, $G_1(\alpha) = G_2(\alpha)$. For $\alpha \in D$, $\alpha = \overline{{}_\lambda r}$, $r \in [v]^\kappa$, $G_1(\alpha) = \overline{q_1}$, $G_2(\alpha) = \overline{q_2}$ where for all $\beta < \kappa$,

$$q_1(\beta) = H_{1_\beta}({}_\lambda r(\beta)) \quad \text{and} \quad q_2(\beta) = H_{2_\beta}({}_\lambda r(\beta)).$$

But ${}_\lambda r(\beta) \in v \subseteq D$ and $\beta <_\lambda r(\beta)$ for all β, so for $\beta \in A$, ${}_\lambda r(\beta) \in B_\beta$ so $H_{1_\beta}({}_\lambda r(\beta)) = H_{2_\beta}({}_\lambda r(\beta))$ and so $q_1 \sim_2 q_2$. This proves the claim.

It can be similarly shown that $\alpha < \beta$ implies $\gamma_\alpha < \gamma_\beta$, so $\{\gamma_\delta\}_{\delta<\kappa_3}$ is an increasing sequence in $[\kappa_2]^{\kappa_2}/\mu_2$.

Claim. *$\{\gamma_\delta\}_{\delta<\kappa_3}$ is cofinal in $[\kappa_2]^{\kappa_2}/\mu_2$.*

Given $r \in [\kappa_2]^{\kappa_2}$, we will find δ such that $\gamma_\delta > r \bmod \mu_2$. We start by considering s = the set of all limit points of r. Since $\mu_2(s) = 1$, let $p \in [\kappa]^\kappa$ be such that $\langle p \rangle_\lambda / {\sim_2} \subseteq s$, and let q be the odd elements of ${}_\lambda p$, i.e., $q = ({}_\lambda p)^1$. Let $H = bk_\kappa(q)$, $\delta = \bar{H}$ in κ_3, and consider γ_δ. By definition, $\gamma_\delta = G \bmod \mu_2$, where $G \in [\kappa_2]^{\kappa_2}$, and for any $\alpha < \kappa_2$, $\alpha = \bar{u}$, $u \in [\kappa]^\kappa$, $G(\alpha) = \bar{v}$, $v \in [\kappa]^\kappa$, where $v(\beta) = H_\beta(u(\beta))$.

Now since $H_\beta(u(\beta) > H_0(u(\beta)) \geq q(u(\beta))$, for all β, this gives us that

$G(\alpha) \geq$ the αth element of $[q]^\kappa/{\sim_2}$ (Lemma 3.1)

$\geq$ the $(\alpha+1)$st element of $[{}_\lambda p]^\lambda/{\sim_2}$

$\geq$ the $(\alpha+1)$st element of s (since $[{}_\lambda p]^\kappa/{\sim_2} \subseteq s$)

$>$ the $(\alpha+1)$st element of r (it is not true in general that the ηth element of s is greater than the ηth element of r, but it is true for successor ordinals)

$>$ the αth element of r.

Since this holds for all α, $\gamma_\delta > r$.

It seems reasonable that this result can be extended in fruitful ways to the problem of $\kappa_n^{\kappa_n}/\mu_m$. The more basic concern, however, seems to be the order-type of $\kappa_2^{\kappa_2}/\mu_2$ itself. It is hoped that the sequence described above can be proved to be all of $\kappa_2^{\kappa_2}$, but this may not be.

3. On beyond κ_ω

The second of Kunen's two theorem cited in Section 2, that $\aleph_1^{\aleph_1}/\mu = \aleph_n$ for some n, for all measures μ on $\aleph_1$, indicates that it would be hopeless to try to get above κ_ω be taking ordinary ultrapowers of κ. The previous section shows further that ultrapowers of the κ_n are likely to be unprofitable too, in that they seem to produce nothing but singular cardinals. In all probability, these ultrapowers are also ultrapowers of κ, and equal to κ_k for some $k < \omega$ indeed, each κ_n can be shown to equal an ultrapower,

$$\kappa^\kappa/\nu_n \text{ where } \nu_n \text{ is the measure defined on } \kappa \text{ by}$$
$$\nu_n(A) = 1 \quad \text{iff } bk_{\kappa^{n-2}}(A) \sim_n bk_{\kappa^{n-2}}(\kappa).$$

All of this suggests that a new method is needed to get a combinatorial hold on cardinals above κ_ω. Such a method is outlined here; it is the ultrapower of κ by a measure on $[\kappa]^\kappa$.

Definition. For $A \subseteq [\kappa]^\kappa$, we define $\nu(A) = 1$ iff for some $p \in [\kappa]^\kappa$, $\langle p \rangle \subseteq A$.

Theorem 3.1. *ν is an $\aleph_1$-additive measure on $[\kappa]^\kappa$.*

Proof. First, suppose $\langle p \rangle \subseteq A$, $\langle q \rangle \subseteq B$ for some p, $q \in [\kappa]^\kappa$, and let $p' \in [p]^\kappa$, $q' \in [q]^\kappa$ be such that for all $\alpha < \kappa$,

$$p'(\alpha) < q'(\alpha) < p'(\alpha + 1).$$

Then $\langle p' \rangle = \langle q' \rangle$ since for every $r \in [p]^\kappa$, there is a corresponding $s \in [q]^\kappa$ such that $r(\alpha) < s(\alpha) < r(\alpha + 1)$ for all α, and so ${}_\omega r = {}_\omega s$, and vice versa. Since $\langle p' \rangle \subseteq \langle p \rangle \subseteq A$ and $\langle q' \rangle \subseteq \langle q \rangle \subseteq B$ we have $\langle p' \rangle \subseteq A \cap B$ and so the measure-one sets form a filter.

Finally, to show additivity, suppose $[\kappa]^\kappa$ is divided up into a countable number of disjoint sets A_n. Let $F: [\kappa]^\kappa \to \omega$ be the partition:

$$F(p) = n \quad \text{where } {}_\omega p \in A_n.$$

By a theorem of [2], $\kappa \to (\kappa)^\kappa + \mathrm{DC}$ implies $\kappa \to (\kappa)^\kappa_\omega$, and hence there is a set $p \in [\kappa]^\kappa$ homogeneous for F. Then if $F''[p]^\kappa = \{n\}$, $\langle p \rangle \subseteq A_n$, and so $\nu(A_n) = 1$.

We can similarly define measures ν_λ and ν_κ using $\langle\ \rangle_\lambda$ and $\langle\ \rangle_\kappa$ and the theorem continues to hold.

Theorem 3.2. $\kappa^{([\kappa]^\kappa)}/\nu \geq \kappa_2$.

Proof. For each $p \in S_2$ we define the partition: $G_p^2: [\kappa]^\kappa \to \kappa$ by

$$G_p^2(q) = p(q(0)).$$

Each G_p^2 is an element of the ultrapower. Suppose we are given $p <_2 q$. Then $A = \{\alpha \mid p(\alpha) < q(\alpha)\}$ is such that $\mu(A) = 1$. Let $x \in [\kappa]^\kappa$ be such that $(x)_\omega \subseteq A$. Then for any $r \in \langle x \rangle$, $G_p^2(r) < G_q^2(r)$. Since $\nu(\langle x \rangle) = 1$, $G_p^2 < G_q^2 \bmod \nu$. Similarly, if $p \sim_2 q$, then $G_p^2 \sim G_q^2 \bmod \nu$.

To prove that the ultrapower is in fact greater than κ_ω requires a combinatorial lemma. Awkward in appearance, it is actually just a beefed-up diagonal intersection property.

Lemma 3.3. *Suppose η is a normal measure on a cardinal λ, $n < \omega$ and Q is any function from $[\lambda]^n$ to 2^λ such that for all $A \in Q''[\lambda]^n$, $\eta(A) = 1$. Then there is a set $B \subseteq \lambda$, $\eta(B) = 1$ such that $\beta \in B$ implies that*

$$\beta \in \bigcap Q''[\beta]^n.$$

Proof. By induction on n.

Theorem 3.4. $\kappa^{([\kappa]^\kappa)}/\nu \geq \kappa_\omega$.

Proof. For $H \in S_n$, let us write: $H(\alpha)$ for H_α, $H(\alpha)(\beta)$ for $[H_\alpha]_\beta$, and $H(\alpha_1)(\alpha_2)\cdots(\alpha_k)$ for $[\cdots[[H_{\alpha_1}]_{\alpha_2}]\cdots]_{\alpha_k}$ and for all $p \in [\kappa]^\kappa$ we define:

$$G_H^n(p) = H(p(0))(p(1))\cdots(p(n-2)).$$

Suppose that $H <_n G$, for some H, $G \in S_n$. Let $A = \{\alpha \mid H(\alpha) <_{n-1} G(\alpha)\}$ and define the maps $Q_i : [\kappa]^i \to 2^\kappa$ for $1 \leq i \leq n-2$ by:

$$Q_1(\beta_1) = \begin{cases} \{\beta \mid H(\beta_1)(\beta) <_{n-2} G(\beta_1)(\beta)\} & \text{if this set is of } \mu\text{-measure-one,} \\ \kappa & \text{otherwise.} \end{cases}$$

and in general,

$$Q_i(\beta_1, \ldots, \beta_i) = \begin{cases} \{\beta \mid H(\beta_1)(\beta_2)\cdots(\beta_i)(\beta) <_{n-i-1} G(\beta_1)(\beta_2)\cdots(\beta_i)(\beta)\} \\ \quad \text{if this set is of } \mu\text{-measure-one,} \\ \kappa \quad \text{otherwise.} \end{cases}$$

(Let $<_1$ be $<$.)

Note that if $H(\beta_1)\cdots(\beta_i) <_{n-i} G(\beta_1)\cdots(\beta_i)$, then $Q_i(\beta_1, \ldots, \beta_i)$ must equal $\{\beta \mid H(\beta_1)\cdots(\beta) <_{n-i-1} G(\beta_1)\cdots(\beta)\}$ by the definition of $<_{n-i}$.

By Lemma 3.3, for each i there is a set A_i such that $\mu(A_i) = 1$ and $\beta \in A_i$, $\beta_1, \ldots, \beta_i < \beta$ imply $\beta \in Q_i(\beta_1, \ldots, \beta_i)$. Let $B = A \cap A_1 \cap \cdots \cap A_{n-2}$. Since $\mu'(B) = 1$, *let* x be such that $(x)_\omega \subseteq B$.

Claim. *G_G^n is greater than G_H^n on $\langle x \rangle$.*

Proof of claim. Suppose $s \in [x]^\kappa$, $t = {}_\omega s$, so that $t(0), t(1), \ldots, t(n-2) \in B$. For all $0 \leq i \leq n-2$, let $\beta_i = F_i({}_\omega s)$. Unwinding our definitions, we find: $\beta_0 \in A$, so

$$Q_1(\beta_0) = \{\beta \mid H(\beta_0)(\beta) <_{n-1} G(\beta_0)(\beta)\}.$$

and since $\beta_1 \in Q_1(\beta_0)$, then

$$Q_2(\beta_0, \beta_1) = \{\beta \mid H(\beta_0)(\beta_1)(\beta) <_{n-3} G(\beta_0)(\beta_1)(\beta)\}$$

$$\vdots$$

then

$$Q_{n-2}(\beta_0, \ldots, \beta_{n-3}) = \{\beta \mid H(\beta_0)\cdots(\beta) <_1 G(\beta_0)\cdots(\beta)\}$$

and

$$\beta_{n-2} \in Q_{n-2}(\beta_0, \ldots, \beta_{n-3})$$

so finally,

$$G_G^n({}_\omega s) = G(\beta_0)\cdots(\beta_{n-2}) > H(\beta_0)\cdots(\beta_{n-2}) = G_H^n({}_\omega s)$$

and the claim is proved. That $H \sim_n G$ implies $G_H^n \sim G_G^n$ is proved in the same manner.

There are a number of scattered results on the ultrapower, for example, assuming AC_λ, $\lambda < \kappa$, $\mathrm{cf}(\kappa^{([\kappa]^\kappa)}/\nu) > \lambda$. For $n < \omega$, the partition $F(p) = p(n)$ can be shown to be the κ_n-th element in the ultrapower, and $F_\omega(p) = \bigcup_{n<\omega} p(n)$ to be the κ_ω-th. One would hope that in the case of AD, $F_{\omega+1}(p) = p(\omega)$ might represent $\aleph_{\omega+1}$, but its cofinality turns out to be $\aleph_2$, while $\aleph_{\omega+1}$ is regular. These results are found in [2].

Further research in this area compares this ultrapower where $\kappa = \aleph_1$ to

$$\aleph_1^{Q_{\aleph_1}(\aleph_1)}/\mu^*$$

where μ^* is a measure on $Q_{\aleph_1}(\aleph_1)$ derived from AD (see [3]).

A deeper analysis shows further that no measure ν on $[\aleph_1]^{\aleph_1}$ can be used to generate a measure on $P_{\aleph_1}(\aleph_1)$.

Another approach to this ultrapower is found in [8].

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